Skeleton relational structures

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Abstract

The theory of skeleton relational structures is very useful in the investigation of the isomorphism between structures in which relations play an important role. It is an important tool for model designers. This theory is also known as lattice theory.

Contents

1 Introduction

Quantum theory deviates in fundamental aspects from classical physics. Quantum theory appears to be ruled by quantum logic, while classical physics is ruled by classical logic.

From contemporary physics we know that elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.

With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the relational structure of the corresponding models. In particular the modularity axiom of the skeleton relational structure differs. That axiom is weakened.

Here we are not interested in quantum logic and classical logic as logic systems. We consider their structure as skeleton relational structures.

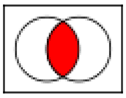
Classical logic is congruent to an orthocomplemented modular lattice.

Quantum logic is congruent to an orthocomplemented weakly modular lattice. Another name for that lattice is orthomodular lattice.

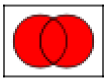
2 Lattices

A subset of the axioms of the skeleton relational structure characterizes it as a half ordered set. A larger subset defines it as a lattice.
A lattice is a set of elements $a, b, c, \ldots$ that is closed for the connections ∩ and ∪.

∩ is called **conjunction**[1]**.**



∪ is called **disjunction.**



These connections obey:

- The set is partially ordered. With each pair of elements a, b belongs an element c, such that $a \subset c$ and $b \subset c$.
- The set is a ∩ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cap b$.
- The set is a ∪ half lattice if with each pair of elements a, b an element c exists, such that $c = a \cup b$.
- The set is a lattice if it is both a ∩ half lattice and a ∪ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{1}$$

$$(a \cap b) \cap c = a \cap (b \cap c) \tag{2}$$

$$a \cap (a \cup b) = a \tag{3}$$

$$a \cup b = b \cup a \tag{4}$$

$$(a \cup b) \cup c = a \cup (b \cup c) \tag{5}$$

$$a \cup (a \cap b) = a \tag{6}$$

The lattice has a partial order inclusion ⊂:

$$a \subset b \Leftrightarrow a \subset b = a \tag{7}$$

A complementary lattice contains two elements n and e with each element a a complementary element a' such that:

[1] http://en.wikipedia.org/wiki/Logical_conjunction

$$a \cap a' = n \quad (8)$$

$$a \cap n = n \quad (9)$$

$$a \cap e = a \quad (10)$$

$$a \cup a' = e \quad (11)$$

$$a \cup e = e \quad (12)$$

$$a \cup n = a \quad (13)$$

e is the unity element; n is the null element of the lattice

An orthocomplemented lattice contains two elements n and e and with each element a an element a'' such that:

$$a \cup a'' = e \quad (14)$$

$$a \cap a'' = n \quad (15)$$

$$(a'')'' = a \quad (16)$$

$$a \subset b \Leftrightarrow b'' \subset a'' \quad (17)$$

2.1 Types of lattices

2.1.1 Distributive lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{18}$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{19}$$

2.1.2 Modular lattice

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \tag{20}$$

2.1.3 Weak modular lattice

A weak modular lattice supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c \\ = a \cup (b \cap c) \cup (d \cap c) \tag{21}$$

where d obeys:

$$(a \cup b) \cap d = d \tag{22}$$

$$a \cap d = n \tag{23}$$

$$b \cap d = n \tag{24}$$

$$[(a \subset g) \; and \; (b \subset g) \Leftrightarrow d \subset g \tag{25}$$

2.1.4 Atomic lattice

In an atomic lattice holds

$$\exists_{p \in L} \forall_{x \in L} \{x \subset p \Rightarrow x = n\} \tag{26}$$

$$\forall_{a \in L} \forall_{x \in L} \{(a < x < a \cap p) \tag{27}$$

$$\Rightarrow (x = a \; or \; x = a \cap p)\}$$

p is an atom

2.1.5 Examples

Both the set of elements of quantum logic and the set of closed subspaces of a separable Hilbert space **Ң** have the structure of an orthomodular lattice. In this respect these sets are congruent.

In quaternionic separable Hilbert space, an atom is an eigensubspace of a corresponding operator. That eigensubspace is spanned by eigenvectors of another operator. The atom specifies a state.
In a complex number based Hilbert space, states exist in pure and in mixed form.

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthomodular lattice. That is an orthocomplented weakly modular and atomic lattice.

The set of closed subspaces of a Hilbert space also has that structure.

2.2 Lattice elements

Lattice elements can, but must not be propositions. In logic systems the elements are considered as propositions. This is why the name “quantum logic” has confused many physicists. For the purpose of model generation the elements of this structure can better be interpreted as modular construction elements.

Thus quantum logic has a treacherous name. It can be considered as a logic system and it can be considered as a skeleton relational structure of a modular system. In this skeleton relational structure the elements can be interpreted as building blocks and as composites of building blocks. With this interpretation the skeleton relational structure can become part of a recipe for modular construction.

2.2.1 Propositions

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false. Propositions take forms like "This is a particle or a wave". In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five logical connectives[2], and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

2.3 Hilbert space

The set of closed subspaces of an infinite dimensional separable Hilbert space is lattice isomorphic with the set of elements of a an orthomodular lattice. This set makes clear that the skeleton relational structure owns other interpretations than the interpretation as a logic system of propositions. What is the added value of the Hilbert space model?

It adds the superposition principle and in the form of Hilbert vectors it shows finer detail than the skeleton relational substructure.

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[2] http://en.wikipedia.org/wiki/Logical_connective

3 Restrictions

The skeleton relational structures can only model countable sets of discrete elements. The structures offer no means for modeling continuums.
The axioms that define these structures specify relations between the elements. These axioms do not specify the content of the elements.

The axioms do not provide a means to implement dynamics. The skeleton relational structures can only model a static status quo.

These restrictions also hold for separable Hilbert spaces.
This does not say that the Hilbert space cannot describe the change of the data that is stored in the eigenspaces of its operators. It means that the control of this change is housed outside the Hilbert space.

4 Hilbert logic

The set of elements of traditional quantum logic is lattice isomorphic with the set of closed subspaces of a separable Hilbert space. However there exist still significant differences between this logic system and the Hilbert space. It is interesting to study where quantum logic differs from this substructure of separable Hilbert spaces.

The gap between the two structures can be closed by refining the specification of quantum logic until it becomes the specification of Hilbert logic.

Step 1: Add to each element as an extra attribute a numeric value that gets the name ***relevance factor***.
Step 2: Require that linear combinations of atomic elements also belong to the new logic system.
Step 3: Introduce the notion of a ***relational coupling measure*** between two linear elements. This measure has properties that are similar to the properties of the inner product of Hilbert space vectors.
Step 4: Close the subsets of the new logic system with respect to this ***relational coupling measure***. This closure adds new elements that are the equivalents of Hilbert vectors. Sets of Hilbert vectors span Hilbert subspaces.
The relevance factor and the relational coupling measure can have values that are taken from a suitable division ring[3]. The resulting logic system will be called Hilbert logic.

4.1 Similarity with Hilbert space

The addition of the relevance factor installs the superposition principle. A linear combination of linear elements is again a linear element.
In this way the Hilbert logic is lattice isomorphic as well topological isomorphic with the corresponding Hilbert space.
Due to this similarity the Hilbert logic will also feature linear operators.
In a Hilbert logic, linear operators can be defined that have linear atoms as their eigen-elements. The eigenspace of these operators is countable. The eigenvalues are numbers that introduce geometry into the model.
Linear elements are the equivalents of Hilbert vectors. General basic modularization structure elements are the equivalents of (closed) subspaces of a Hilbert space.
The measure of the relational coupling between two linear elements is the equivalent of the inner product between two Hilbert vectors.

This intermezzo merely explains that the only difference between quantum logic and the set of closed subspaces of a separable Hilbert space is the superposition principle.

4.2 Norm

The modulus of the relevance factor can be used to define the notion of the norm of the element.

4.3 Interdependent

Two elements are interdependent when their relative relevance factor is non-zero.

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[3] The restriction to a division ring is taken from the fact that also Hilbert space restricts its numbers to elements of a division ring.

4.4 Dimension

The elements of Hilbert logic have a dimension. It is the number of mutually independent normed elements that span the element.

4.5 Free elements

The relevance factor can be used to define the notion of free elements.
If the modulus of the relevance factor is maximized at a fixed value, for example unity, then a free element can be defined as an element for which this maximum is reached. It means that free elements cannot be considered as members of a superposition. On the other hand free elements can be superpositions of bounded elements.
When used in this way the relevance factor takes the role of a ***probability amplitude***.

4.6 Binding building blocks and encapsulating their relations.

The superposition principle is the driving force behind the binding of building blocks into composites. Composites are construction elements that can be written as linear combinations of more basic building blocks.

In fact the composites are spanned by vectors that are elements of the constituents.

Thus, the constituents of composites are not free elements. Instead they are bounded elements. The constituents lose their individuality and the relations in which these constituents play a role become less apparent. With other words, the superposition principle installs the relational encapsulation of the constituents of the composite.

Part of the binding is due to the embedding of the Hilbert subspaces and their content in corresponding subspaces of the Gelfand triple. This will be treated later.